Lecture notes : Recent (and not that recent) forcing techniques on finite support iterations (Iterated Forcing Theory and Cardinal Invariants)

Mejía, Diego Alejandro

数理解析研究所講究録
第2081巻 2018年 7-27

Lecture notes: Recent (and not that recent) forcing techniques on finite support iterations

Diego Alejandro Mejía

Faculty of Science, Shizuoka University.
836 Ohya, Suruga-ku, 422-8529 Shizuoka, Japan.
diego.mejia@shizuoka.ac.jp

Abstract

These are the lectures notes of the minicourse of three sessions presented by the author in the RIMS 2017 Set Theory Workshop on Iterated Forcing and Cardinal Invariants. We focus on the ideas about how to use forcing techniques on FS iteration to produce models where several cardinal invariants are pairwise different. These techniques will converge at the end in a sketch of the proof of the consistency of a constellation of Cichoń's diagram with 10 different values (the maximum possible) modulo *three strongly compact cardinals.*

1 Introduction

This text is the lecture notes of the minicourse delivered in the RIMS 2017 Set Theory Workshop on Iterated Forcing and Cardinal Invariants. In this minicourse, we reviewed some forcing techniques in the context of FS (finite support) iterations with applications on combinatorics of the real line. We focused in techniques of matrix iterations and coherent systems from [BS89, BF11, Mej13, FFMM18, Mej], and of FS iteration with ultrafilters from [She00, GMS16, BCM]. These are useful techniques to force that several cardinal invariants of the continuum are pairwise different.

The order of this lecture notes is similar to the order of the lectures. In Section 2 we review a general way to define cardinal invariants through *relational systems.* Section 3 introduces the technique of coherent systems of FS iterations and offers general tools to force many different values for cardinal invariants of the continuum. In the last section, we present the technique of FS iterations with ultrafilters and its applications, concretely, we sketch very roughly Kellner, Goldstern and Shelah's [GKS] argument to construct a ccc poset, using *four strongly compact cardinals*, to force that Cichoń's diagram can be divided into 10 different values (the maximum possible). By combining the techniques of

2D coherent systems with ultrafilters along iterations, we refine this result and show that *three strongly compact cardinals* are enough.

Throughout this text we do not intend to give particular details on the proofs of the results, we focus instead on the general ideas to carry out the arguments.

We fix some notation. If b is a function with domain ω, denote by $\prod b := \prod_{i<\omega} b(i)$ and, when $h : \omega \to \omega$ denote $\mathcal{S}(b,h) := \prod_{i<\omega}[b(i)]^{\leq h(i)}$. Depending on the context, an ordinal η may represent the constant sequence η of length ω. For any set A, id_A denotes the identity function on A.

Concerning forcing notions, $\mathbb{C}_A$ denotes the poset of finite partial function from $A \times \omega$ into ω ordered by $\supseteq$, i.e., it is the poset that adds a family of Cohen reals indexed by A. Cohen forcing is denoted by $\mathbb{C} := \mathbb{C}_1$. Hechler's poset that adds a dominating real is denoted by $\mathbb{D}$. The standard σ-centered poset that adds an eventually different real is defined as $\mathbb{E} := \omega^{<\omega} \times \bigcup_{m<\omega} \mathcal{S}(\omega, m)$ (in $\mathcal{S}(\omega, m)$ both ω and m denote constant functions of length ω) ordered by $(t,\psi) \leq (s,\varphi)$ iff $s \subseteq t$, $\forall i < \omega(\varphi(i) \subseteq \psi(i))$ and $t(i) \notin \varphi(i)$ for any $i \in |t| \smallsetminus |s|$.

2 Cardinal invariants of the continuum

In many cases, cardinal invariants of the continuum are defined through *relational systems.*

Definition 2.1. A *relational system* is a triplet $\mathbf{A} = \langle X, Y, \sqsubset \rangle$ where $\sqsubset$ is a relation contained in $X \times Y$. For $x \in X$ and $y \in Y$, $x \sqsubset y$ is often read y *$\sqsubset$-dominates* x. A family $F \subseteq X$ is $\mathbf{A}$-*unbounded* if there is no real in Y that $\sqsubset$-dominates every member of F. Dually, $D \subseteq Y$ is an $\mathbf{A}$-*dominating* family if every member of X is $\sqsubset$-dominated by some member of D. The cardinal $\mathfrak{b}(\mathbf{A})$ denotes the least size of an $\mathbf{A}$-unbounded family and $\mathfrak{d}(\mathbf{A})$ is the least size of an $\mathbf{A}$-dominating family.

The relational system $\mathbf{A}^\perp := \langle Y, X, \not\sqsupset \rangle$ is referred to as the *dual of* $\mathbf{A}$. Note that $\mathfrak{b}(\mathbf{A}^\perp) = \mathfrak{d}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}^\perp) = \mathfrak{b}(\mathbf{A})$.

The cardinal invariants $\mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A})$ may not always exist. Concretely, $\mathfrak{b}(\mathbf{A})$ does not exist iff $\mathfrak{d}(\mathbf{A}) = 1$. Dually, $\mathfrak{d}(\mathbf{A})$ does not exists iff $\mathfrak{b}(\mathbf{A}) = 1$.

Inequalities between cardinal invariants are often proved using the *Tukey order* between relational systems. If $\mathbf{A} = \langle X, Y, \sqsubset \rangle$ and $\mathbf{A}' = \langle X', Y', \sqsubset' \rangle$ are relational systems, $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{A}'$ means that there are two maps $\varphi : X \to X'$ and $\psi : Y' \to Y$ such that, for any $x \in X$ and $y' \in Y'$, $\varphi(x) \sqsubset' y'$ implies $x \sqsubset \psi(y')$. In this case, the ψ-image of any $\mathbf{A}'$-dominating set is $\mathbf{A}$-dominating, and the φ-image of any $\mathbf{A}$-unbounded set is $\mathbf{A}'$-unbounded, thus $\mathfrak{b}(\mathbf{A}') \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq \mathfrak{d}(\mathbf{A}')$. Say that $\mathbf{A}$ and $\mathbf{A}'$ are *Tukey equivalent*, denoted by $\mathbf{A} \cong_{\mathrm{T}} \mathbf{A}'$, if $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{A}'$ and $\mathbf{A}' \preceq_{\mathrm{T}} \mathbf{A}$.

Example 2.2. (1) Consider $\omega^\omega = \langle \omega^\omega, \omega^\omega, \leq^* \rangle$ where $x \leq^* y$ iff $x(i) \leq y(i)$ for all but finitely many $i < \omega$. Define $\mathfrak{b} := \mathfrak{b}(\omega^\omega)$ and $\mathfrak{d} := \mathfrak{d}(\omega^\omega)$.

(2) Let X be a set and $\mathcal{I}$ a family of subsets of X such that

(i) if $B \in \mathcal{I}$ and $A \subseteq B$ then $A \in \mathcal{I}$,

(ii) $[X]^{<\aleph_0} \subseteq \mathcal{I}$, and

(iii) $X \notin \mathcal{I}$.

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r, dashed] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u, dashed] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r, dashed] \arrow[u, dashed] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 1: Cichoń's diagram. The arrows represent $\leq$. The dashed arrows mean $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{M})\}$.

Consider the relational systems $\mathcal{I} = \langle \mathcal{I}, \mathcal{I}, \subseteq \rangle$ and $\mathbf{Cv}(\mathcal{I}) := \langle X, \mathcal{I}, \in \rangle$ ([1]). Note that $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$, $\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$, $\mathfrak{b}(\mathbf{Cv}(\mathcal{I})) = \mathrm{non}(\mathcal{I})$ and $\mathfrak{d}(\mathbf{Cv}(\mathcal{I})) = \mathrm{cov}(\mathcal{I})$. We refer to these four cardinals as the *cardinal invariants associated with* $\mathcal{I}$. Of utter importance in combinatorics of the real line are the cardinal invariants associated with $\mathcal{M}$, the ideal of meager subsets of the reals, and those associated with $\mathcal{N}$, the ideal of Lebesgue measure zero subsets of the reals.

(3) For two functions x and φ with domain ω, define $x \in^* \varphi$ by $\forall^\infty i < \omega(x(i) \in \varphi(i))$, which is read φ *localizes* x.

Let b be a function with domain ω such that $b(i) \neq \emptyset$ for all $i < \omega$, and let $h \in \omega^\omega$. Consider the relational system $\mathbf{Lc}(b, h) := \langle \prod b, \mathcal{S}(b, h), \in^* \rangle$ and put $\mathfrak{b}^{\mathrm{Lc}}_{b,h} := \mathfrak{b}(\mathbf{Lc}(b, h))$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} := \mathfrak{d}(\mathbf{Lc}(b, h))$, which are often referred to as *localization cardinals*.

(4) For two functions x and φ with domain ω, define $x \in^\infty \varphi$ by $\exists^\infty i < \omega(x(i) \in \varphi(i))$. Denote its negation by $\varphi \not\ni^* x$, which is read φ *anti-localizes* x.

Let b and h be as in the previous item. Consider the relational system $\mathbf{aLc}(b, h) := \langle \mathcal{S}(b, h), \prod b, \not\ni^* \rangle$.Define $\mathfrak{b}^{\mathrm{aLc}}_{b,h} := \mathfrak{b}(\mathbf{aLc}(b, h))$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h} := \mathfrak{d}(\mathbf{aLc}(b, h))$, which are referred to as *anti-localization cardinals.* Note that $\mathbf{aLc}(b, h)^\perp = \langle \prod b, \mathcal{S}(b, h), \in^\infty \rangle$.

(5) For two functions x and y with domain ω, define $x \neq^* y$ by $\forall^\infty i < \omega(x(i) \neq y(i))$, which is read x *and* y *are eventually different.*

For b as in the previous item, consider the relational system $\mathbf{Ed}(b) := \langle \prod b, \prod b, \neq^* \rangle$. It is easy to see that $\mathbf{Ed}(b) \cong_{\mathrm{T}} \mathbf{aLc}(b, 1)$, so $\mathfrak{b}(\mathbf{Ed}(b)) = \mathfrak{b}^{\mathrm{aLc}}_{b,1}$ and $\mathfrak{d}(\mathbf{Ed}(b)) = \mathfrak{d}^{\mathrm{aLc}}_{b,1}$.

The cardinals presented in (1) and (2) form part of the very well-known *Cichoń's diagram*, which is illustrated in Figure 1. It is well-known that this diagram is *complete* in the sense that no other inequality can be proved between two cardinal invariants in there. See e.g. [BJ95] for details and original references.

The interesting cases of the localization and anti-localization cardinals are when each $b(i)$ is at most countable and $h(i) < |b(i)|$ for all but finitely many $i < \omega$. When h does not go to infinity, $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$ is a natural number (that depends on h) and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} = \mathfrak{c}$ (see [GS93, Lemma 1.11]). A similar result holds for the anti-localization cardinals when $\frac{h(i)}{|b(i)|}$ does not converge to 0 (see [CM, Sect. 3]).

[1] Here, $X = \bigcup \mathcal{I}$, so we can say that $\mathbf{Cv}(\mathcal{I})$ only depends on $\mathcal{I}$.

Some cardinal invariants associated with $\mathcal{N}$ and $\mathcal{M}$ can be characterized by $\mathfrak{b}$, $\mathfrak{d}$ and the localization and anti-localization cardinals. For this, consider the following cardinal invariants:

$$\begin{array}{rlrl} \mathrm{minLc} := & \min\{\mathfrak{b}^{\mathrm{Lc}}_{b,\mathrm{id}_\omega} : b \in \omega^\omega\}, & \mathrm{supLc} := & \sup\{\mathfrak{d}^{\mathrm{Lc}}_{b,\mathrm{id}_\omega} : b \in \omega^\omega\}, \\ \mathrm{supaLc} := & \sup\{\mathfrak{b}^{\mathrm{aLc}}_{b,1} : b \in \omega^\omega\}, & \mathrm{minaLc} := & \min\{\mathfrak{d}^{\mathrm{aLc}}_{b,1} : b \in \omega^\omega\}. \end{array}$$

In the definition of minLc and supLc, if id_ω is replaced by any $h \in \omega^\omega$ that goes to infinity, then the cardinal does not change. Likewise for supaLc and minaLc when 1 is replaced by any $h \geq^* 1$ in ω^ω (see [CM, Sect. 3]).

Theorem 2.3. *Let $h \in \omega^\omega$.*

(a) (Bartoszyński [BJ95, Thm. 2.3.9]) *If h goes to infinity then* $\mathrm{add}(\mathcal{N}) = \mathfrak{b}^{\mathrm{Lc}}_{\omega,h}$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}^{\mathrm{Lc}}_{\omega,h}$.

(b) (Bartoszyński [BJ95, Lemmas 2.4.2 and 2.4.8]) *If $h \geq^* 1$ then* $\mathfrak{b}^{\mathrm{aLc}}_{\omega,h} = \mathrm{non}(\mathcal{M})$ *and* $\mathfrak{d}^{\mathrm{aLc}}_{\omega,h} = \mathrm{cov}(\mathcal{M})$.

(c) [CM, Sect. 3] $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}, \mathrm{minLc}\}$ *and* $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}, \mathrm{supLc}\}$.

(d) (Miller [Mil81]) $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{minaLc}\}$ *and* $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{supaLc}\}$

In relation with the cardinal invariants of Cichoń's diagram, for $b, c : \omega \to (\omega+1) \setminus \{0\}$, $g, h \in \omega^\omega$ such that $g <^* b$, g diverges to infinity, $h \geq^* 1$ and $\frac{h(i)}{c(i)}$ converges to 0, the cardinals $\mathfrak{b}^{\mathrm{Lc}}_{b,g}$, $\mathfrak{b}^{\mathrm{aLc}}_{c,h}$, minLc and supaLc lie between $\mathrm{add}(\mathcal{N})$ and $\mathrm{non}(\mathcal{M})$ (left side of the diagram), and $\mathfrak{d}^{\mathrm{Lc}}_{b,g}$, $\mathfrak{d}^{\mathrm{aLc}}_{c,h}$, supLc and minaLc lie between $\mathrm{cov}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{N})$ (right side of the diagram). This is because:

(1) If $|b(i)| \leq |b'(i)|$ for all but finitely many $i < \omega$, and $h' \leq^* h$, then $\mathbf{Lc}(b,h) \preceq_{\mathrm{T}} \mathbf{Lc}(b',h')$ and $\mathbf{aLc}(b',h') \preceq_{\mathrm{T}} \mathbf{aLc}(b,h)$.

(2) If $b \in \omega^\omega$ then $\mathbf{Lc}(b, b-1) \cong_{\mathrm{T}} \mathbf{aLc}(b,1)$.

In addition, if $\sum_{i<\omega} \frac{h(i)}{c(i)} < \infty$ then $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{aLc}^{\perp}(c,h)$, so $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}^{\mathrm{aLc}}(c,h)$ and $\mathfrak{d}^{\mathrm{aLc}}(c,h) \leq \mathrm{non}(\mathcal{N})$. On the other hand, if $\sum_{i<\omega} \frac{h(i)}{c(i)} = \infty$ then $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{aLc}(d,1) \preceq_{\mathrm{T}} \mathbf{aLc}(c,h)$ where $d(i) := \lceil \frac{c(i)}{h(i)} \rceil$, so $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}(c,h)$ and $\mathfrak{b}(c,h) \leq \mathrm{non}(\mathcal{N})$.

Kellner and Shelah [KS12] constructed an ω^ω-bounding proper poset that forces continuum many pairwise different cardinals of the form $\mathfrak{b}^{\mathrm{aLc}}_{c,h}$ and continuum many of the form $\mathfrak{d}^{\mathrm{Lc}}_{b,g}$; Kamo and Osuga [KO14] constructed a ccc poset using a FS iteration to force infinitely many pairwise different cardinals of the form $\mathfrak{b}^{\mathrm{aLc}}_{c,1}$, moreover, by using a weakly inaccessible cardinal, such a poset can be constructed to force continuum many values of those cardinals. With other methods, Brendle and the author [BM14] constructed a ccc poset using FS iterations to force infinitely many values of cardinals of the form $\mathfrak{b}^{\mathrm{Lc}}_{b,g}$, and continuum many modulo a weakly inaccessible from the ground model.

Question 2.4. Is it consistent with ZFC (without using inaccessible cardinals) that there are continuum many pairwise different cardinals of the form $\mathfrak{b}^{\mathrm{aLc}}_{c,1}$? can we get a similar consistency result for cardinals of the form $\mathfrak{b}^{\mathrm{Lc}}_{b,g}$?

Question 2.5. Is it consistent with ZFC that there are continuum many pairwise different cardinals of the form $\mathfrak{d}^{\mathrm{aLc}}_{c,h}$ (even just infinitely many)?

Very recently, Cardona and the author [CM] merged the techniques from [KO14] and [BM14] to construct a FS iteration of ccc posets that forces infinitely many pairwise different cardinals of both types $\mathfrak{b}^{\mathrm{Lc}}_{b,g}$ and $\mathfrak{b}^{\mathrm{aLc}}_{c,1}$, even continuum many of the latter type modulo a weakly inaccessible from the ground model. However, this poset puts the cardinals of the form $\mathfrak{b}^{\mathrm{Lc}}_{b,g}$ below those of the form $\mathfrak{b}^{\mathrm{aLc}}_{c,1}$.

Question 2.6. Is it consistent with ZFC that there are infinitely many pairwise different cardinals of the form $\mathfrak{b}^{\mathrm{Lc}}_{b,g}$ and $\mathfrak{b}^{\mathrm{aLc}}_{c,1}$ such that they appear in some alternate order?

Details and more facts about localization and anti-localization cardinals can be found in [GS93, KS12, KO14, CM].

3 Coherent systems of FS iterations

There are several number of simple and sophisticated techniques to construct ccc iterations with finite support, which are useful when dealing with three or more cardinal invariants at the same time[2]. For instance, Blass and Shelah [BS89] discovered a way to construct a FS iteration through a two-dimensional arrangement of posets (and generic extensions, see Figure 2), technique that was used and improved in [BF11, Mej13, FFMM18, Mej] to force several different values to cardinal invariants of the continuum. In particular, in [FFMM18] appears the first practical example of a FS iteration through a three-dimensional arrangement of ccc posets (see Figure 3), and the formalization of such type of constructions, now known as *coherent systems of FS iterations.*

This section is dedicated to introduce the general notion of coherent systems of FS iterations and to show how to construct such systems in order to have some desired effect on cardinal invariants of the continuum. In the applications, we do not use three-dimensional constructions, in its place we use a stronger coherent system from [Mej] that can be seen as a two-dimensional coherent system that allows vertical support restrictions.

Definition 3.1. Let M be a transitive model of ZFC. When $\mathbb{P} \in M$ and $\mathbb{Q}$ are posets, say that $\mathbb{P}$ *is a complete subposet of* $\mathbb{Q}$ *with respect to* M, abbreviated $\mathbb{P} \lessdot_M \mathbb{Q}$, if $\mathbb{P}$ is a subposet of $\mathbb{Q}$ and any maximal antichain of $\mathbb{P}$ that belongs to M is still a maximal antichain in $\mathbb{Q}$.

If in addition N is another transitive model of ZFC, $M \subseteq N$ and $\mathbb{Q} \in N$, then $\mathbb{P} \lessdot_M \mathbb{Q}$ implies that, whenever G is $\mathbb{Q}$-generic over N, $G \cap \mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G \cap \mathbb{P}] \subseteq N[G]$ (see Figure 4).

Example 3.2. Let $M \subseteq N$ be transitive models of ZFC. When $\mathbb{P} \in M$ it is clear that $\mathbb{1} \lessdot_M \mathbb{P}$ and $\mathbb{P} \lessdot_M \mathbb{P}$. Also, if $\mathbb{S}$ is a Suslin ccc poset then $\mathbb{S}^M \lessdot_M \mathbb{S}^N$.

Definition 3.3 ([FFMM18, Def. 3.2]). A *coherent system (of FS iterations)* $\mathbf{s}$ is composed by the following objects:

[2]Countable support iterations of proper forcing are not useful in this case because they either are restricted to force $\mathfrak{c} \leq \aleph_2$ or they collapse ω_2 after $\omega_2 + \omega_1$ many steps.

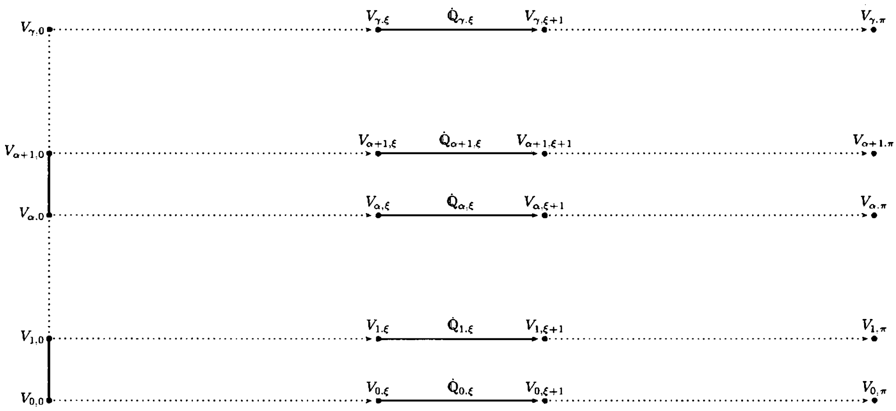

Figure 2: 2D coherent system.

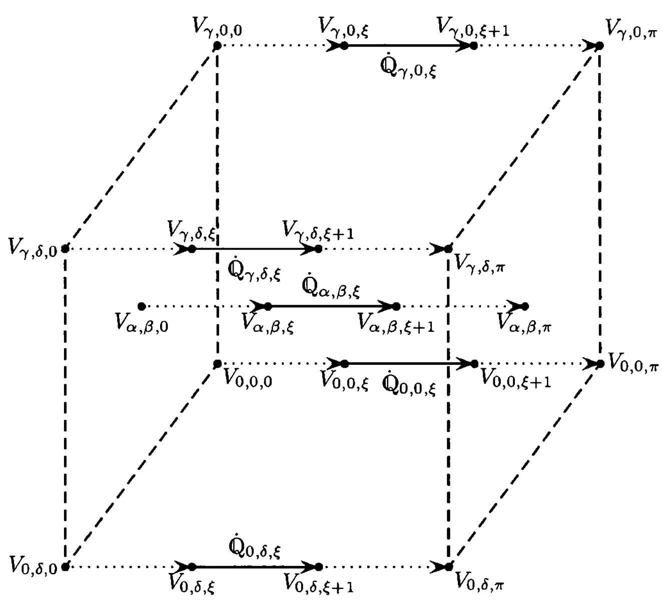

Figure 3: 3D coherent system.

$$\begin{tikzcd} N \arrow[r, "\mathbb{Q}"'] & N[G] \\ M \arrow[r, "\mathbb{P}"'] & M[G \cap \mathbb{P}] \end{tikzcd}$$

Figure 4: Generic extensions of pairs of posets ordered like $\mathbb{P} \lessdot_M \mathbb{Q}$.

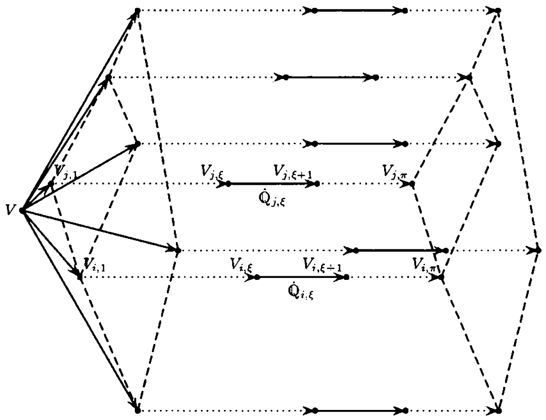

Figure 5: Coherent system of FS iterations. The figures in dashed lines represent the 'shape' of the partial order $\langle I, \leq \rangle$.

(1) a partially ordered set I^{s}, an ordinal π^{s}, and

(2) for each $i \in I^{\mathrm{s}}$, a FS iteration $\mathbb{P}^{\mathrm{s}}_{i,\pi^{\mathrm{s}}} = \langle \mathbb{P}^{\mathrm{s}}_{i,\xi}, \dot{\mathbb{Q}}^{\mathrm{s}}_{i,\xi} : \xi < \pi^{\mathrm{s}} \rangle$ such that, for any $i \leq j$ in I and $\xi < \pi^{\mathrm{s}}$, if $\mathbb{P}^{\mathrm{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathrm{s}}_{j,\xi}$ then $\mathbb{P}^{\mathrm{s}}_{j,\xi}$ forces $\dot{\mathbb{Q}}^{\mathrm{s}}_{i,\xi} \lessdot_{V^{\mathbb{P}^{\mathrm{s}}_{i,\xi}}} \dot{\mathbb{Q}}^{\mathrm{s}}_{j,\xi}$.

According to this notation, $\mathbb{P}^{\mathrm{s}}_{i,0}$ is the trivial poset and $\mathbb{P}^{\mathrm{s}}_{i,1} = \dot{\mathbb{Q}}^{\mathrm{s}}_{i,0}$. We often refer to $\langle \mathbb{P}^{\mathrm{s}}_{i,1} : i \in I^{\mathrm{s}} \rangle$ as the *base of the coherent system* s. Condition (2) implies that $\mathbb{P}^{\mathrm{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathrm{s}}_{j,\xi}$ whenever $i \leq j$ in I^{s} and $\xi \leq \pi^{\mathrm{s}}$.

For $j \in I^{\mathrm{s}}$ and $\eta \leq \pi^{\mathrm{s}}$ we write $V^{\mathrm{s}}_{j,\eta}$ for the $\mathbb{P}^{\mathrm{s}}_{j,\eta}$-generic extensions. Concretely, when G is $\mathbb{P}^{\mathrm{s}}_{j,\eta}$-generic over V, $V^{\mathrm{s}}_{j,\eta} := V[G]$ and $V^{\mathrm{s}}_{i,\xi} := V[\mathbb{P}^{\mathrm{s}}_{i,\xi} \cap G]$ for all $i \leq j$ in I^{s} and $\xi \leq \eta$. Note that $V^{\mathrm{s}}_{i,\xi} \subseteq V^{\mathrm{s}}_{j,\eta}$ and $V^{\mathrm{s}}_{i,0} = V$ (see Figure 5).

We say that the coherent system s has the *ccc* if, additionally, $\mathbb{P}^{\mathrm{s}}_{i,\xi}$ forces that $\dot{\mathbb{Q}}^{\mathrm{s}}_{i,\xi}$ has the ccc for each $i \in I^{\mathrm{s}}$ and $\xi < \pi^{\mathrm{s}}$. This implies that $\mathbb{P}^{\mathrm{s}}_{i,\xi}$ has the ccc for all $i \in I^{\mathrm{s}}$ and $\xi \leq \pi^{\mathrm{s}}$.

We consider the following particular cases.

(i) When I^{s} is a well-ordered set, we say that s is a *2D-coherent system (of FS iterations)* (see Figure 2).

(ii) If $I^{\mathbf{s}} = \gamma^{\mathbf{s}} \times \delta^{\mathbf{s}}$ where $\gamma^{\mathbf{s}}$ and $\delta^{\mathbf{s}}$ are ordinals and the order of $I^{\mathbf{s}}$ is defined as $(\alpha, \beta) \leq (\alpha', \beta')$ iff $\alpha \leq \alpha'$ and $\beta \leq \beta'$, we say that $\mathbf{s}$ is a *3D-coherent system (of FS iterations)* (see Figure 3).

For a coherent system $\mathbf{s}$ and a set $J \subseteq I^{\mathbf{s}}$, $\mathbf{s}|J$ denotes the coherent system with $I^{\mathbf{s}|J} = J$, $\pi^{\mathbf{s}|J} = \pi^{\mathbf{s}}$ and with its FS iterations defined as for $\mathbf{s}$; if $\eta \leq \pi^{\mathbf{s}}$, $\mathbf{s}{\restriction}\eta$ denotes the coherent system with $I^{\mathbf{s}{\restriction}\eta} = I^{\mathbf{s}}$, $\pi^{\mathbf{s}{\restriction}\eta} = \eta$ and with the iterations defined up to η as for $\mathbf{s}$. Note that, if $i_0 \in I^{\mathbf{s}}$, then $\mathbf{s}|\{i_0\}$ is just the FS iteration $\mathbb{P}^{\mathbf{s}}_{i,\pi^{\mathbf{s}}} = \langle \mathbb{P}^{\mathbf{s}}_{i,\xi}, \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} : \xi < \pi^{\mathbf{s}} \rangle$.

The upper indices $\mathbf{s}$ are omitted when there is no risk of ambiguity.

It is very common to work with coherent systems where I has a maximum i^*, this to ensure that there is an actual final generic extension of the construction. Note that this final model $V_{i^*,\pi}$ is obtained by the FS iteration $\langle \mathbb{P}_{i^*,\xi}, \dot{\mathbb{Q}}_{i^*,\xi} : \xi < \pi \rangle$ on the top. The iterations constructed below the iteration at i^* help to deal with its combinatorics.

Typically, a coherent system of FS iterations $\mathbf{s}$ is constructed by transfinite recursion. Concretely, $\mathbf{s}{\restriction}\xi$ is constructed by recursion on $\xi \leq \pi$ as follows. In the step $\xi = 0$, we determine the partial order $\langle I, \leq \rangle$ that will support the base of the coherent system; in the limit step it is just enough to take direct limits; for the successor step, assuming that $\mathbf{s}{\restriction}\xi$ has been constructed, the system $\langle \dot{\mathbb{Q}}_{i,\xi} : i \in I \rangle$ of names of posets ($\dot{\mathbb{Q}}_{i,\xi}$ is a $\mathbb{P}_{i,\xi}$-name) that will determine how it is forced in stage ξ is determined, and afterwards $\mathbf{s}{\restriction}(\xi+1)$ is defined so that it extends $\mathbf{s}{\restriction}\xi$ and $\mathbb{P}_{i,\xi+1} = \mathbb{P}_{i,\xi} * \dot{\mathbb{Q}}_{i,\xi}$ for each $i \in I$. To have that $\mathbf{s}{\restriction}(\xi+1)$ is indeed a coherent system, we require that $\Vdash_{\mathbb{P}_{j,\xi}} \dot{\mathbb{Q}}_{i,\xi} \lessdot_{V_{i,\xi}} \dot{\mathbb{Q}}_{j,\xi}$ whenever $i \leq j$ in I.

In this text, we concentrate only on ccc coherent systems that use the following type of iterands for any successor step $\xi \to \xi+1$ with $\xi > 0$. The first step of the iteration, that is, the construction of $\langle \mathbb{P}_{i,1} : i \in I \rangle$ (recall that $\mathbb{P}_{i,1} = \dot{\mathbb{Q}}_{i,0}$) receives a special treatment and it is used to add generic reals in the 'vertical direction'. This will be discussed starting from Remark 3.7.

(I) **Adding a full generic real.** This means that, from step ξ to $\xi+1$, we add a real that is generic (e.g. random, dominating) over $V_{i,\xi}$ for any $i \in I$. For example, this is done by using a Suslin ccc poset $\mathbb{S}_\xi$ that adds the desired type of real and force with $\dot{\mathbb{Q}}_{i,\xi} = \mathbb{S}_\xi^{V_{i,\xi}}$ at stage ξ for all $i \in I$.

(II) **Adding a restricted generic (real) over a small set.** Let θ be a cardinal, $\Delta(\xi) \in I$ and let $\dot{N}_\xi$ be a $\mathbb{P}_{\Delta(\xi),\xi}$-name of a transitive model of (some finite fragment of) ZFC of size $< \theta$. Here, we add a generic (real) over $\dot{N}_\xi$, concretely, if $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name of a poset that belongs to $\dot{N}_\xi$ such that $\mathbb{P}_{i,\xi}$ forces that $\dot{\mathbb{Q}}_\xi$ is ccc for all $i \geq \Delta(\xi)$, the we put

$$\dot{\mathbb{Q}}_{i,\xi} := \begin{cases} \dot{\mathbb{Q}}_\xi & \text{if } i \geq \Delta(\xi), \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

This is indeed a particular case of (III) below, but here the generic object is more restricted. For example, if $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for $\mathbb{D}^{\dot{N}_\xi}$, the real it adds is dominating over $\dot{N}_\xi$ (not necessarily over $V_{\Delta(\xi),\xi}$). In applications, the cardinal θ is small with respect of some relevant information in the iteration (e.g., below the cofinality of the length of the FS iterations).

(III) **Adding a vertically restricted generic.** From step ξ to $\xi+1$, we add a generic over $V_{\Delta(\xi),\xi}$ for some chosen $\Delta(\xi) \in I$. For instance, if $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name of a poset that is forced to be ccc by $\mathbb{P}_{i,\xi}$ for all $i \geq \Delta(\xi)$, $\dot{\mathbb{Q}}_{i,\xi}$ is defined as in (II).

Now, we discuss the effect of such coherent systems on cardinal invariants of the form $\mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A})$ for some relational system $\mathbf{A}$. In the context of FS iterations, these cardinals are dealt with by using a strong type of unbounded and dominating families.

Definition 3.4. Let $\mathbf{A} = \langle X, Y, \sqsubset\rangle$ be a relational system and let θ be a cardinal number.

(1) When M is a set and $x \in X$, say that x *is* $\mathbf{A}$*-unbounded over* M iff $x \not\sqsubset y$ for any $y \in Y \cap M$.

(2) A subset F of X is a *strongly* θ*-*$\mathbf{A}$*-unbounded family* if $|F| \geq \theta$ and, for any $y \in Y$, $|\{x \in F : x \sqsubset y\}| < \theta$.

(3) A subset D of Y is a *strongly* θ*-*$\mathbf{A}$*-dominating family* if there is a $<\theta$-directed partial order $\langle K, \trianglelefteq\rangle$ such that $D = \{a_k : k \in K\}$ and, for any $x \in X$, there is some $k_0 \in K$ such that $x \sqsubset a_k$ for all $k \trianglerighteq k_0$ in L.

Lemma 3.5. *(a) If $F \subseteq X$ is strongly θ-$\mathbf{A}$-unbounded then $\mathfrak{b}(\mathbf{A}) \leq |F|$ and, when θ is regular, $|F| \leq \mathfrak{d}(\mathbf{A})$.*

(b) If $D \subseteq Y$ is strongly θ-$\mathbf{A}$-dominating then $\theta \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq |D|$.

Note that, when θ is regular, any strongly θ-$\mathbf{A}$-unbounded family is θ-$\mathbf{A}^\perp$-dominating. Indeed, if $F \subseteq X$ is a strongly θ-$\mathbf{A}$-unbounded family, then we can find a one-to-one enumeration $F := \{x_\alpha : \alpha < \gamma\}$ for some ordinal γ that has cofinality θ, so γ with the usual order is $<\theta$-directed and witnesses that F is strongly θ-$\mathbf{A}^\perp$-dominating.

When forcing values of $\mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A})$ with FS iterations of ccc posets, inequalities as in Lemma 3.5(b) can be obtained in a simpler way than using a strong dominating family. The need of using strong dominating families is present when applying the Boolean ultrapower technique from [KTT, GKS]. This will be discussed in Section 4.

The following result is a tool to deal with (names of) reals within coherent systems.

Lemma 3.6 ([Mej, Lemma 2.4]). *Let θ be an uncountable regular cardinal. Assume that* $\mathbf{s}$ *is a coherent system that satisfies:*

(i) I has a maximum i^* and $I \smallsetminus \{i^*\}$ is $<\theta$-directed,

(ii) each $\mathbb{P}_{i,\xi}$ forces that $\dot{\mathbb{Q}}_{i,\xi}$ is θ-cc, and

(iii) for any $\xi < \pi$, if $\mathbb{P}_{i^*,\xi}$ is the direct limit of $\langle \mathbb{P}_{i,\xi} : i < i^*\rangle$ then $\mathbb{P}_{i^*,\xi}$ forces that $\dot{\mathbb{Q}}_{i^*,\xi} = \bigcup_{i<i^*} \dot{\mathbb{Q}}_{i,\xi}$.

Then, for any $\xi \leq \pi$,

(a) $\mathbb{P}_{i^*,\xi}$ is the direct limit of $\langle \mathbb{P}_{i,\xi} : i < i^*\rangle$ and

(b) if $\gamma < \theta$ and $\dot{f}$ is a $\mathbb{P}_{i^*,\xi}$-name of a function from γ into $\bigcup_{i<i^*} V_{i,\xi}$ then $\dot{f}$ is (forced to be equal to) a $\mathbb{P}_{i,\xi}$-name for some $i < i^*$. In particular, the reals in $V_{i^*,\xi}$ are precisely the reals in $\bigcup_{i<i^*} V_{i,\xi}$.

Let $\theta \leq \mu$ and κ be uncountable regular cardinals, ν and λ cardinals such that $\kappa \leq \mathrm{cf}(\nu)$ and $\mu, \nu \leq \lambda = \lambda^{<\theta}$, and let I be a partial order that has a maximum element i^*. Put $I^- := I \smallsetminus \{i^*\}$. Assume that $\mathbf{A} = \langle X, Y, \sqsubset \rangle$ is an *analytic* relational system (in the sense that X and Y are analytic subsets of some Polish spaces $\mathbb{R}_1$ and $\mathbb{R}_2$, and $\sqsubset$ is analytic in $\mathbb{R}_1 \times \mathbb{R}_2$). We show, in a general way, how to construct coherent systems that forces each of $\mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A})$ take one of the values θ, κ, μ, ν and λ. Let $\pi = \lambda\nu\mu$ (ordinal product). The idea is to construct ccc coherent systems of FS iterations of length π on I by using the rules (I)-(III) above. Note that $\mathrm{cf}(\pi) = \mu$, so μ is the cofinality of the length of the iterations. Also, if reals are added at each step of the iteration, it is forced that $\mathfrak{c} = \lambda$. The tools (A)-(C) below can be used simultaneously for different relational systems.

(A) Force $\mathfrak{d}(\mathbf{A}) \leq \mu \leq \mathfrak{b}(\mathbf{A})$ (and even equality) by adding full generic reals

At cofinally many steps $\xi < \pi$ use ccc posets to add full generic $\mathbf{A}$-dominating reals (according to (I)). If $\dot{D}$ is the (name of the) family of those dominating reals, then $\mathbb{P}_{i^*,\pi}$ forces that $\dot{D}$ is strongly μ-$\mathbf{A}$-dominating, so $\mathfrak{d}(\mathbf{A}) \leq |\dot{D}|$ and $\mu \leq \mathfrak{b}(\mathbf{A})$. In fact $\mathfrak{d}(\mathbf{A}) \leq \mu$ because $\dot{D}$ can be defined from a cofinal subset of π of size μ.

Recall that any FS iteration adds Cohen real al limit steps, so it is forced that $\mathrm{non}(\mathcal{M}) \leq \mu \leq \mathrm{cov}(\mathcal{M})$. On the other hand, if $\mathbf{Cv}(\mathcal{M}) \preceq_{\mathrm{T}} \mathbf{A}$ can be proved in a definable way, then $\mathbb{P}_{i^*,\pi}$ forces that $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$.

For example, random forcing can be used to add a full generic $\mathbf{Cv}(\mathcal{N})^\perp$-dominating real (i.e., a full random real), so it will be forced that $\mathrm{non}(\mathcal{N}) \leq \mu \leq \mathrm{cov}(\mathcal{N})$, even more, equality holds since $\mathbf{Cv}(\mathcal{M}) \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{N})^\perp$ (by Rothberger, see [BJ95, Lemma 2.1.7]).

(B) Force $\theta \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq \lambda$ by adding restricted generic reals over small sets

Here we further assume that either $I = \{i^*\}$ or that I^- is non-empty of size $\leq \lambda$ and $< \theta$-directed. We work in the second case. By Lemma 3.5, it is enough to force the existence of a strongly θ-$\mathbf{A}$-dominating family of size λ. At each interval of the form $[\lambda\rho, \lambda(\rho+1))$ for $\rho < \nu\mu$, we fix a subset C_ρ of size λ of that interval, which we use to add restricted $\mathbf{A}$-generic reals over sets of size $< \theta$ (according to (II)). Fix a bijection $g_\rho = (g^0_\rho, g^1_\rho) : C_\rho \to I^- \times \lambda$. For each $i \in I^-$, construct a family $\{\dot{N}_{\rho,i,\zeta} : \zeta < \rho\}$ of $\mathbb{P}_{i,\lambda\rho}$-names of transitive models of ZFC of size $< \theta$ such that

(i) for any $B \subseteq \bigcup_{\varrho<\rho} C_\varrho$ of size $< \theta$ there is some upper bound $\eta \in C_\rho$ of B such that $g^0_\rho(\eta)$ is above $\{g^0_\varrho(\xi) : \xi \in B \cap C_\varrho,\ \varrho < \rho\}$ and $\mathbb{P}_{g^0_\rho(\eta),\lambda\rho}$ forces that
$$\bigcup_{\varrho<\rho}\ \bigcup_{\xi\in B\cap C_\varrho} \dot{N}_{\varrho,g_\varrho(\xi)} \subseteq \dot{N}_{\rho,g_\rho(\xi)};$$

(ii) for any $\mathbb{P}_{i^*,\lambda\rho}$-name $\dot{x}$ of a member of X there is some $\xi \in C_\rho$ such that $\dot{x}$ is a $\mathbb{P}_{g^0_\rho(\xi),\lambda\rho}$-name and $\mathbb{P}_{g^0_\rho(\xi),\lambda\rho}$ forces that $\dot{x} \in \dot{N}_{\rho,g_\rho(\xi)}$.

Note that Lemma 3.6 should be used for this construction. For each $\xi \in C_\rho$ use a $\mathbb{P}_{g^0_\rho(\xi),\lambda\rho}$-name $\dot{\mathbb{Q}}_\xi$ of a poset in $\dot{N}_{\rho,g_\rho(\xi)}$ that adds an $\mathbf{A}$-dominating real $\dot{y}_\rho$ over $\dot{N}_{\rho,g_\rho(\xi)}$ and such that $\mathbb{P}_{i^*,\xi}$ forces that $\dot{\mathbb{Q}}_\xi$ is ccc. Put $C := \bigcup_{\rho<\nu\mu} C_\rho$. If $\dot{\mathbb{Q}}_{i,\xi}$ is defined as in (II) for any

$i \in I$ and $\xi \in C$, then $\mathbb{P}_{i^*,\pi}$ forces that $\dot{E} := \{\dot{y}_\xi : \xi \in C\}$ is a strongly θ-$\mathbf{A}$-dominating family where C is ordered by $\xi \trianglelefteq \eta$ iff $\xi \leq \eta$, $g^0_\varrho(\xi) \leq g^0_\rho(\eta)$ where ϱ, ρ are the (unique) ordinals such that $\xi \in C_\varrho$ and $\eta \in C_\rho$, and $\mathbb{P}_{g^0_\rho(\eta)}$ forces that $\dot{N}_{\varrho,g_\varrho(\xi)} \subseteq \dot{N}_{\rho,g_\rho(\eta)}$ (by (i), this order is $<\theta$-directed). Therefore, as C has size λ, $\mathbb{P}_{i^*,\pi}$ forces that $\theta \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq \lambda$.

Note that the inequality $\mathfrak{d}(\mathbf{A}) \leq \lambda$ would follow trivially since $\mathfrak{d}(\mathbf{A}) \leq |Y| \leq \mathfrak{c}$ and $\mathbb{P}_{i^*,\pi}$ forces $\mathfrak{c} = \lambda$. On the other hand, the previous construction could be simplified just to force $\theta \leq \mathfrak{b}(\mathbf{A})$ without using a strongly dominating family. So, why do we get into the pain of constructing a strongly dominating family? Such family is useful when applying Boolean ultrapowers to the constructed ccc forcing. A further explanation is presented in Section 4.

How about the converse inequalities? A tool to deal with this is Judah and Shelah [JS90] and Brendle [Bre91] preservation theory of strongly κ-$\mathbf{A}$-unbounded families along finite support iterations. Though this topic was not dealt with in the minicourse, we briefly mention the general idea of its application. This demands more conditions in $\mathbf{A}$, e.g., X is a Polish perfect space and $\{x \in X : x \sqsubset y\}$ is meager in X for any $y \in Y$, which implies that $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{M})$ (in a very definable way) and that Cohen reals (in X) are $\mathbf{A}$-unbounded over the ground model. Hence, for any uncountable regular cardinal κ, any FS iteration of length κ of non-trivial ccc posets forces that the Cohen reals added at limit stages of the iteration form a strongly κ-$\mathbf{A}$-unbounded family of size κ. When considering iterations longer than κ, the preservation theory is used to guarantee that the strongly unbounded family of Cohen reals added at step κ is preserved strongly unbounded until the end of the iteration. To apply this theory to our present discussion, it is required that the iterands $\dot{\mathbb{Q}}_{i^*,\xi}$, for any $\xi < \pi$, are forced to be θ-$\mathbf{A}$-*good* (the main notion of the preservation theory). In this case, it follows that, for any uncountable regular $\kappa \in [\theta, \pi]$, the strongly κ-$\mathbf{A}$-unbounded family added by $\mathbb{P}_{i^*,\kappa}$ is preserved in the $\mathbb{P}_{i^*,\pi}$-generic extension, so $\mathbb{P}_{i^*,\pi}$ forces that $\mathfrak{b}(\mathbf{A}) \leq \kappa \leq \mathfrak{d}(\mathbf{A})$. As κ was arbitrary, $\mathfrak{b}(\mathbf{A}) \leq \theta$ and $\lambda \leq \mathfrak{d}(\mathbf{A})$ (so equality is forced). Details on the preservation theory can be found, e.g., in [CM, Sect. 4].

Case $I = \{i^*\}$. The construction can be easily modified to get a strongly dominating family. The preservation theory also applies.

(C) Force $\min\{\kappa, \mu\} \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq \max\{\mu, \nu\}$ by adding vertically restricted generic reals

Here we further assume that I^- is $<\kappa$-directed and has size $\leq \nu$. Fix a function $t : \nu\mu \to I^-$ such that, for any $\varrho < \nu\mu$ and $i \in I^-$ there is some $\varrho \leq \rho < \nu\mu$ such that $t(\rho) \geq i$ (it can be constructed because $|I| \leq \nu$). For each $\rho < \nu\mu$ choose a point $\eta_\rho \in [\lambda\rho, \lambda(\rho+1))$ and use a $\mathbb{P}_{t(\rho),\eta_\rho}$-name $\dot{\mathbb{Q}}_{\eta_\rho}$ of a poset that adds a $\mathbf{A}$-dominating real $\dot{z}_\rho$ over $V_{t(\rho),\eta_\rho}$ and such that $\mathbb{P}_{i^*,\eta_\rho}$ forces that $\dot{\mathbb{Q}}_{\eta_\rho}$ is ccc. If $\dot{\mathbb{Q}}_{i,\eta_\rho}$ is defined as in (III) then $\mathbb{P}_{i^*,\pi}$ forces that $\dot{H} := \{\dot{z}_\rho : \rho < \nu\mu\}$ is a strongly $\min\{\kappa,\mu\}$-$\mathbf{A}$-dominating family where $\nu\mu$ is ordered by $\varrho \trianglelefteq \rho$ iff $\varrho \leq \rho$ and $t(\varrho) \leq t(\rho)$. Thus, $\mathbb{P}_{i^*,\pi}$ forces $\min\{\kappa,\mu\} \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq |\nu\mu| = \max\{\mu,\nu\}$.

How about the converse inequalities? Assume that $J \subseteq I^-$ is a strongly κ-I^--unbounded set and that, for each $j \in J$, $\mathbb{P}_{j,1}$ adds a real $\dot{x}_j$ in X such that $\mathbb{P}_{i^*,1}$ forces that $\dot{x}_j$ is $\mathbf{A}$-unbounded over $V_{i,1}$ for any $i \in I$ with $j \not\leq i$, and $\dot{x}_j \neq \dot{x}_{j'}$ whenever $j \neq j'$.

Here, the preservation theory of unbounded reals from [BS89, BF11, Mej13] can be used to preserve $\dot{x}_j$ to be $\mathbf{A}$-unbounded over $V_{i,\pi}$ for any $i \in I$ with $j \not\leq i$. This ensures that $\dot{F} = \{\dot{x}_j : j \in J\}$ is a strongly κ-$\mathbf{A}$-unbounded family in the final extension, so $\mathfrak{b}(\mathbf{A}) \leq |J| \leq \mathfrak{d}(\mathbf{A})$. Hence, if I^- contains such a strongly unbounded subset of size κ then it is forced that $\mathfrak{b}(\mathbf{A}) \leq \kappa$; if I^- contains such a strongly unbounded subset of size ν then it is forced that $\nu \leq \mathfrak{d}(\mathbf{A})$.

The preservation of unbounded reals is guaranteed when, for each $0 < \xi < \pi$ where a full generic real is added, a Suslin ccc poset $\mathbb{S}_\xi$ is used in that step and, for each $i \in I^-$, $\mathbb{P}_{i,\xi}$ forces that $\mathbb{S}^{V_{i,\xi}}$ is $\mathbf{A}$-good (notion mentioned in (B)). Here we need that $\mathbf{A}$ is a Polish relational system, so $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{M})$ and, as $\mathbb{P}_{i^*,\pi}$ adds μ-cofinally many Cohen reals that form a strongly μ-$\mathbf{A}$-unbounded family, it forces $\mathfrak{b}(\mathbf{A}) \leq \mu \leq \mathfrak{d}(\mathbf{A})$. Details can be found in, e.g., [CM, Sect. 4].

There are some other ways to add full generic reals without using Suslin ccc posets, for example, according to [BS89, BF11], if I is a well-ordered set, by recursion on $i \in I$ it can be constructed a $\mathbb{P}_{i,\xi}$-name $\dot{U}_{i,\xi}$ of a non-principal ultrafilter on ω such that, for any $i \leq j$ in I, $\mathbb{P}_{j,\xi}$ forces $\dot{U}_{i,\xi} \subseteq \dot{U}_{j,\xi}$ and, when $j = i+1$, $\mathbb{M}(\dot{U}_{i,\xi}) \lessdot_{V_{i,\xi}} \mathbb{M}(\dot{U}_{i+1,\xi})$. Even more, they can be constructed so that unbounded reals in ω^ω can be preserved (in the aforementioned way) and also to ensure preservation of certain type of mad families.

Remark 3.7. Note that (A) and (B) (as well as (I) and (II)) fit in the context of a single FS iteration. The importance of the partial order I and the base $\langle \mathbb{P}_{i,1} : i \in I\rangle$ of a coherent system is related to (C), which allows to force additional values of cardinal characteristics of the continuum. Furthermore, when equalities are obtained in (C), it is necessary that $\mathrm{cf}(\nu) \geq \kappa$ since $\mathrm{cf}(\mathfrak{d}(I^-)) = \mathfrak{b}(I^-)$, $\mathfrak{b}(I^-) = \kappa$ and $\mathfrak{d}(I^-) = \nu$.

The following example presents bases of coherent systems that are suitable for (C).

Example 3.8. (1) Let ν be an uncountable regular cardinal and $I = \nu+1$ with the usual order. Put $\mathbb{P}_{i,1} := \mathbb{C}_{\lambda+i}$. With $\kappa = \nu$, $I^- = \nu$ is $<\nu$-directed, $J = \{i+1 : i \in I^-\}$ is strongly ν-I^--unbounded and each $\mathbb{P}_{i+1,1}$ adds a Cohen real $\dot{c}_i$ over $V_{i,1}$, which is also $\mathbf{A}$-unbounded over the same model for any Polish relational system $\mathbf{A}$. It is clear that the demands on (C) for I and $\langle \mathbb{P}_{i,1} : i \in I\rangle$ are satisfied (even those for the converse inequalities).

(2) Let κ be an uncountable regular cardinal and let ν be a cardinal such that $\nu = \nu^{<\kappa}$. Consider $I = [\nu]^{<\kappa} \cup \{\nu\}$ ordered by $\subseteq$. It is clear that $I^- = [\nu]^{<\kappa}$ is $<\kappa$-directed, $i^* := \max(I) = \nu$ and that $|I| = \nu$. For each $A \in I$ define $\mathbb{P}_{A,1} := \mathbb{C}_A$. Note that, for any $L \subseteq \nu$ of size $\geq \kappa$, the set $J_L := \{\{\alpha\} : \alpha \in L\}$ is strongly κ-I^--unbounded. Moreover, if $\dot{c}_\alpha$ denotes a Cohen real added by $\mathbb{C}_{\{\alpha\}}$ for any $\alpha \in \nu$, then it is forced to be Cohen over $V_{A,1}$ for any $A \in I^-$ such that $\alpha \notin A$. In particular, J_κ and J_ν are as required to force the equalities in (C).

Now, we are ready to give some applications.

Theorem 3.9 ([BCM]). *Let $\theta_0 \leq \theta_1 \leq \theta_2 \leq \mu \leq \nu$ be uncountable regular cardinals and let λ be a cardinal such that $\nu \leq \lambda = \lambda^{<\theta_2}$. Then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \theta_2$, $\mathrm{non}(\mathcal{M}) = \mu$, $\mathrm{cov}(\mathcal{M}) = \nu$ and $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.*

First part of the proof. Construct a coherent system on $I = \nu+1$ of FS iterations of length $\pi := \lambda\nu\mu$ such that $\mathbb{P}_{i,1} = \mathbb{C}_{\lambda+i}$ (as in Example 3.8(1)) and the further iterands are determined by (II) and (III) according to the following criteria:

(i) add restricted amoeba reals over sets of size $< \theta_0$ according to (II) and (B) for $\theta = \theta_0$;

(ii) add restricted random reals over sets of size $< \theta_1$ according to (II) and (B) for $\theta = \theta_1$;

(iii) add restricted Hechler dominating reals over sets of size $< \theta_2$ according to (II) and (B) for $\theta = \theta_2$;

(iv) add vertically restricted eventually different reals in ω^ω (with $\mathbb{E}$) according to (III) and (C) for $\kappa = \nu$.

By (C) it follows that $\mathbb{P}_{\nu,\pi}$ forces $\mu \leq \mathfrak{b}(\mathbf{Ed}(\omega)) = \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\mathbf{Ed}(\omega)) \leq \nu$. Actually, equality holds because the μ-cofinally many Cohen reals added at the limit steps in $\mathbb{P}_{\nu,\pi}$ form a strongly μ-$\mathbf{Cv}(\mathcal{M})$-unbounded family (so $\mathrm{non}(\mathcal{M}) \leq \mu$), and because no full generic reals are added along this iteration.

The methods of (B) indicate that $\mathbb{P}_{\nu,\pi}$ forces $\theta_0 \leq \mathrm{add}(\mathcal{N})$, $\theta_1 \leq \mathrm{cov}(\mathcal{N})$, $\theta_2 \leq \mathfrak{b}$ and $\mathfrak{d}, \mathrm{non}(\mathcal{N}) \leq \lambda = \mathfrak{c}$. With the exception of $\mathfrak{b}$ and $\mathfrak{d}$, equalities can be guaranteed by Judah-Shelah and Brendle preservation theory. To force $\mathfrak{b} \leq \theta_2$ and $\lambda \leq \mathfrak{d}$, we need a very special argument that is explained in Section 4. $\square$

The following result is an strengthening of [FFMM18, Thm. 5.4] by allowing that the cardinal ν is singular. Though [FFMM18, Thm. 5.4] was proved using a 3D coherent system, the result below uses a coherent system with a base as in Example 3.8(2). In both results, it can be forced in addition that $\mathfrak{a} = \kappa$ (the almost-disjointness number).

Theorem 3.10 ([Mej]). *Let $\theta_0 \leq \theta_1 \leq \kappa \leq \mu$ be uncountable regular cardinals and let ν, λ be cardinals such that $\mu \leq \nu = \nu^{<\kappa} \leq \lambda = \lambda^{<\theta_1}$. Then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \nu$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.*

Proof. Construct a coherent system on $I = [\nu]^{<\kappa} \cup \{\nu\}$ of FS iterations of length $\pi := \lambda\nu\mu$ such that $\mathbb{P}_{A,1} = \mathbb{C}_A$ (as in Example 3.8(2)) and the further iterands are determined by (I)-(III) according to the following criteria:

(i) add full eventually different reals in ω^ω according to (I) and (A);

(ii) add restricted amoeba reals over sets of size $< \theta_0$ according to (II) and (B) for $\theta = \theta_0$;

(iii) add restricted random reals over sets of size $< \theta_1$ according to (II) and (B) for $\theta = \theta_1$;

(iv) add vertically restricted Hechler dominating reals according to (III) and (C).

The arguments in (A)-(C) guarantee that $\mathbb{P}_{\nu,\pi}$ forces the desired statements. $\square$

The coherent system of the previous proof can be extended directly to a coherent system on $\mathcal{P}(\nu)$ ordered by $\subseteq$. This can be seen as a 2D coherent system of dimensions $\nu \times \pi$ that allows restriction over any subset of the vertical component, i.e., when $A \subseteq \nu$ and $\xi \leq \pi$, $\mathbb{P}_{A,\xi}$ is understood as the vertical restriction of $\mathbb{P}_{\nu,\xi}$ on A. See Figure 6 for a graphic interpretation.

Horizontal restriction over an arbitrary subset of a FS iteration has been considered for quite some time. The more general framework is known as *template iterations*, a

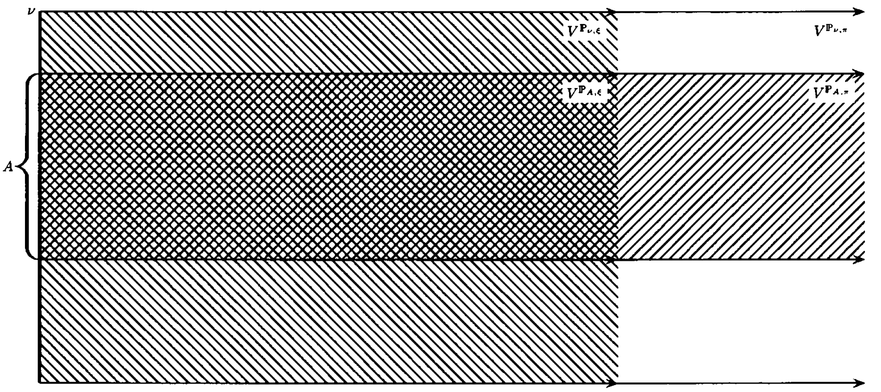

Figure 6: Matrix iteration with vertical support restrictions.

technique created by Shelah [She04] to construct a ccc poset that forces $\aleph_1 < \mathfrak{d} < \mathfrak{a}$. This is a general method to construct an iteration over a linear order L with respect to a well-founded system $\mathcal{I}$ of subsets of I (e.g., for a typical FS iteration $L = \delta$ is an ordinal and $\mathcal{I} = \delta + 1$). This allows to construct a system $\langle \mathbb{P}_A : A \subseteq L \rangle$ such that $\mathbb{P}_A \lessdot \mathbb{P}_B$ whenever $A \subseteq B \subseteq L$, that is, $\mathbb{P}_A$ is understood as the restriction of $\mathbb{P}_L$ on A. Details, improvements an applications can be found in e.g. [Bre02, Bre03, Bre05, Mej15, FM17].

In theory, we can construct 2D coherent systems that allow restriction on arbitrary rectangles. However, new results using this method are unknown at the moment.

4 FS iterations with ultrafilters

We deal with the problem of forcing $\mathfrak{b} \leq \kappa$ and $\lambda \leq \mathfrak{d}$ in the proof of Theorem 3.9. There, the iterands of the FS iteration $\langle \mathbb{P}_{\nu,\xi}, \dot{\mathbb{Q}}_{\nu,\xi} : \xi < \pi \rangle$ have size $< \kappa$ with the exception of the iterands used to add restricted eventually different reals. These small iterands behave well with the preservation theory for $\langle \omega^\omega, \leq^* \rangle$, but this cannot be ensured for the restricted $\mathbb{E}$. Although $\mathbb{E}$ is ω^ω-good, restricted versions of it may add dominating reals, for instance, Pawlikowski [Paw92] showed that a proper ω^ω-bounding Silver type poset forces that both $\mathbb{E}^V$ and random forcing restricted to the ground model add dominating reals. Judah and Shelah [JS93] constructed a two step iteration of the form $\mathbb{C} * \mathbb{T}$, where $\mathbb{T}$ is a Suslin ccc poset coded in the ground model V, that forces that random forcing restricted to $V^{\mathbb{T}}$ adds a dominating real.

This difficulty, in the case of random forcing, was attacked by Shelah [She00] to prove the consistency of $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$. Under suitable hypothesis, he forced with $\mathbb{C}_{\omega_\omega}$ followed by a FS iteration of length $\omega_{\omega+1}$ of restricted random forcing (in a more sophisticated way than the one presented in (II) and (III) of Section 3). As $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}$ implies that $\mathrm{cov}(\mathcal{N})$ has uncountable cofinality (Bartoszyński [Bar88]), it is necessary that the iterands in the previous iteration do not add dominating reals. In fact, this was dealt with by the construction, along de iteration, of sequences of finitely additive measures on $\mathcal{P}(\omega)$ that help to control that the restricted random posets do not add dominating reals.

A weaker variation of this technique was used to prove the following precursor of Theorem 3.9.

Theorem 4.1 ([GMS16]). *Assume that $\theta_0 \leq \theta_1 \leq \theta_2 \leq \mu = \mu^{\aleph_0}$ are regular uncountable cardinals and λ is cardinal such that $\mu < \lambda = \lambda^{<\theta_2}$. If $\lambda \leq 2^\mu$ then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \theta_2$, $\mathrm{non}(\mathcal{M}) = \mu$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$.*

The requirement $\lambda \leq 2^\mu$ concerns the application of the following topological result.

Theorem 4.2 (Engelking and Karłowicz [EK65]). *If $\mu = \mu^{\aleph_0}$ and $\delta < (2^\mu)^+$ then there is a set $H \subseteq \mu^\delta$ of size $\leq \mu$ such that any countable partial function from δ into μ is extended by some member of H.*

For motivational purposes, we briefly explain the forcing construction to prove Theorem 4.1. It consists of a FS iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \delta \rangle$ of length $\delta = \lambda\mu$ where the first θ_2 iterands are Cohen forcing $\mathbb{C}$ and, afterwards, ccc posets of size $< \theta_2$ are used as in (i)-(iii) of the proof Theorem 3.9, and also restricted $\mathbb{E}$ of size $\leq \mu$. Let $E \subseteq \delta$ be the set of steps where we use restricted $\mathbb{E}$. This set should be defined in such a way that $\mathbb{P}_\delta$ forces that the set $\{\dot{e}_\xi : \xi \in E\}$ of restricted eventually different reals form a strongly μ-$\mathbf{Ed}(\omega)$-dominating family of size λ. A method similar to (B) of Section 3 helps to deal with this strongly dominating family, so the actual problem is to ensure that the iteration forces $\mathfrak{b} \leq \theta_2$, and for that we need to be careful in the way thc rcstrictions of $\mathbb{E}$ are defined.

Before we continue this discussion, it is crucial to understand why $\mathbb{E}$ does not add dominating families, even more, how $\mathbb{E}$ preserves strongly ω^ω-unbounded families. We first fix some notation. For $(s,m) \in \omega^{<\omega} \times \omega$ define $\mathbb{E}(s,m) := \{s\} \times \mathcal{S}(\omega, m)$. Note that $\mathbb{E} = \bigcup_{(s,m)\in\omega^{<\omega}\times\omega} \mathbb{E}(s,m)$. When D is a non-principal ultrafilter on ω and $\langle \varphi_n : n < \omega \rangle$ is a sequence of slaloms in $\mathcal{S}(\omega, m)$, define the *D-limit of* $\langle \varphi_n : n < \omega \rangle$ by $\varphi = \lim_n^D \varphi_n$ if, for any $i, j < \omega$, $j \in \varphi(i)$ iff $\{n < \omega : j \in \varphi_n(i)\} \in D$. Note that this D-limit is in $\mathcal{S}(\omega, m)$. If $\bar{p} = \langle p_n : n < \omega \rangle$ is a sequence in $\mathbb{E}(s,m)$, $p_n = (s, \varphi_n)$, the *D-limit of* $\bar{p}$ is defined by $\lim_n^D p_n := (s, \lim_n^D \varphi_n)$. For any poset $\mathbb{P}$ and any sequence $\bar{p} = \langle p_n : n < \omega \rangle$, denote by $\dot{W}_{\mathbb{P}}(\bar{p}) := \{n < \omega : p_n \in \dot{G}_{\mathbb{P}}\}$ where $\dot{G}_{\mathbb{P}}$ is the canonical name of the generic set. The subindex $\mathbb{P}$ is omitted when clear from the context.

Lemma 4.3. *Let D be a non-principal ultrafilter on ω. If G is $\mathbb{E}$-generic over V then, in $V[G]$, D can be extended to an ultrafilter in D^* on $\mathcal{P}(\omega) \cap V[G]$ such that, for any $(s,m) \in \omega^{<\omega} \times \omega$ and any sequence $\bar{p} \in \mathbb{E}(s,m) \cap V$ that has its D-limit in G, $\dot{W}(\bar{p})[G] \in D^*$.*

This lemma is a direct consequence of the following claim in V.

Claim 4.4. *Assume $N < \omega$, $\{(s_k, m_k) : k < N\} \subseteq \omega^{<\omega} \times \omega$, $\{\bar{p}^k : k < N\}$ such that each $\bar{p}^k = \langle p_{k,n} : n < \omega \rangle$ is a sequence in $\mathbb{E}(s_k, m_k)$, q_k is the D-limit of $\bar{p}^k$ for each $k < N$, and $q \in \mathbb{E}$ is stronger than every q_k. If $a \in D$ then q forces that $a \cap \bigcap_{k<N} \dot{W}(\bar{p}^k)$ is non-empty.*

Proof. We can express the forcing conditions as $p_{k,n} = (s_k, \varphi_{k,n})$, $q_k = (s_k, \varphi_k)$ where each φ_k is the D-limit of $\langle \varphi_{k,n} : n < \omega \rangle$ in $\mathcal{S}(\omega, m_k)$. Assume that $q' = (t, \psi) \leq q$ in $\mathbb{E}$. As

$q' \leq q_k$, by the definition of φ_k we have $\{n < \omega : \forall i \in |t| \smallsetminus |s_k|(t(i) \notin \varphi_{k,n}(i))\} \in D$. Hence $a \cap \bigcap_{k<N}\{n < \omega : \forall i \in |t| \smallsetminus |s_k|(t(i) \notin \varphi_{k,n}(i))\}$ is no-empty, so choose an n in that set. Put $r := (t, \psi')$ where $\psi'(i) := \psi(i) \cup \bigcup_{k<N} \varphi_{k,n}(i)$, which is clearly a condition stronger than q' and $q_{n,k}$ for any $k < N$, so it forces $n \in a \cap \bigcap_{k<N} \dot{W}(\bar{p}^k)$. □

The effect of the previous lemma in the preservation of strongly unbounded families in ω^ω can be generalized thanks to the following notion. Denote by $\mathrm{Fr} := \{x \subseteq \omega : |\omega \smallsetminus x| < \aleph_0\}$ the *Frechet filter*. A filter on ω is *free* iff it contains Fr. Denote by F^+ the family of subsets of ω that intersect every member of F.

Definition 4.5 ([BCM]). Let $\mathbb{P}$ be a poset, F a free filter on ω and let μ be an infinite cardinal.

(1) A set $Q \subseteq \mathbb{P}$ is *F-linked* if, for any sequence $\bar{p} = \langle p_n : n < \omega \rangle$ in Q, there exists a $q \in \mathbb{P}$ that forces $\dot{W}(\bar{p}) \in F^+$.

(2) A set $Q \subseteq \mathbb{P}$ is *ultrafilter-linked*, abbreviated *uf-linked*, if Q is D-linked for any non-principal ultrafilter D on ω.

(3) The poset $\mathbb{P}$ is *μ-F-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} P_\alpha$ for some sequence $\langle P_\alpha : \alpha < \mu \rangle$ of F-linked subsets of $\mathbb{P}$. The notion *μ-uf-linked* is defined likewise. When $\mu = \aleph_0$, we write *σ-F-linked* and *σ-uf-linked*.

(4) The poset $\mathbb{P}$ is *μ-F-Knaster* if any subset of $\mathbb{P}$ of size μ contains an F-linked set of size μ. The notion *μ-uf-Knaster* is defined likewise.

When $F \subseteq F'$ are free filters, it is clear that any F-linked set is F'-linked. In particular, a set is uf-linked iff it is F-linked for every free filter F. Though Fr-linked is the weakest, and uf-linked is the strongest among these properties, they are equivalent for some posets. In the result below, $\mathfrak{p}$ denotes the *pseudo-intersection number*, which is the smallest cardinal where Martin's axiom for σ-centered posets fails.

Lemma 4.6 ([Mej, Lemma 5.5]). *Let $\mathbb{P}$ be a poset.*

(a) If F is a free filter on ω generated by $< \mathfrak{p}$-many sets, then any subset of $\mathbb{P}$ is F-linked iff it is Fr*-linked.*

(b) If $\mathbb{P}$ has $\mathfrak{p}$-cc then any subset of $\mathbb{P}$ is uf-linked iff it is Fr*-linked.*

Lemma 4.3 implies that $\mathbb{E}$ is σ-uf-linked, and hence $\aleph_1$-uf-Knaster. So the preservation of strongly unbounded families is a consequence of the following result.

Theorem 4.7 ([BCM]). *If κ is an uncountable regular cardinal then any κ-*Fr*-Knaster poset preserves all the strongly κ-ω^ω-unbounded families from the ground model.*

Proof. Let $\mathbb{P}$ be a κ-Fr-Knaster poset and let $F \subseteq \omega^\omega$ be a κ-strongly unbounded family in the ground model. Towards a contradiction, assume that there is a $\mathbb{P}$-name $\dot{h}$ of a real in ω^ω and a $p \in \mathbb{P}$ such that $p \Vdash |\{x \in F : x \leq^* \dot{h}\}| \geq \kappa$. Find $F' \subseteq F$ of size κ, a family of conditions $\{p_x : x \in F'\} \subseteq \mathbb{P}$ and a natural number m such that, for each $x \in F'$, $p_x \leq p$ and $p_x \Vdash \forall n \geq m(x(n) \leq \dot{h}(n))$. As $\mathbb{P}$ is κ-Fr-Knaster, there is some $F'' \subseteq F'$ of size κ such that $\{p_x : x \in F''\}$ is Fr-linked.

Note that there is a $j \geq m$ such that the set $\{x(j) : x \in F''\}$ is infinite. (otherwise F'' would be bounded, which contradicts that F is strongly κ-unbounded). Choose $\{x_n : n < \omega\} \subseteq F''$ such that $x_n(j) \neq x_{n'}(j)$ whenever $n \neq n'$. For each $n < \omega$, put $p_n := p_{x_n}$. As $\bar{p} = \langle p_n : n < \omega \rangle$ is a sequence in a Fr-linked set, there is a condition $q \in \mathbb{P}$ such that $q \Vdash$"$\dot{W}(\bar{p})$ is infinite". Therefore, q forces that $\exists^\infty n < \omega(x_n(j) \leq \dot{h}(j))$, which is a contradiction. $\square$

We continue our discussion of the construction for Theorem 4.1. As $\mathbb{P}_{\theta_2} = \mathbb{C}_{\theta_2}$ already adds a strongly θ_2-ω^ω-unbounded family, by Theorem 4.7 it would be enough to construct the iteration in such a way that $\mathbb{P}_\delta/\mathbb{P}_{\theta_2}$ is θ_2-Fr-Knaster. The posets of size $< \theta_2$ are ϑ-uf-linked for some $\vartheta < \theta_2$, so they can be used without worry. We actually need to be very careful on how to define the steps for restricted $\mathbb{E}$.

For the moment, assume that the iteration has been constructed such that, for each $\xi \in E$, there is a complete subposet $\mathbb{P}'_\xi$ of $\mathbb{P}_\xi$ of size $\leq \mu$ and $\dot{\mathbb{Q}}_\xi = \mathbb{E}^{V^{\mathbb{P}_\xi}}$. Let us try to prove that $\mathbb{P}_\delta$ is θ_2-Fr-Knaster. Assume that $\langle p_\alpha : \alpha < \theta_2 \rangle$ is a sequence of members of $\mathbb{P}_\delta$. Wlog assume that for each $\alpha < \theta_2$ and $\xi \in E \cap \operatorname{dom} p_\alpha$, $p_\alpha(\xi)$ is a $\mathbb{P}'_\xi$-name and $p_\alpha{\restriction}\xi$ decides the pair $(s^\alpha_\xi, m^\alpha_\xi)$ such that $p_\alpha(\xi) \in \mathbb{E}(s^\alpha_\xi, m^\alpha_\xi)$. By the Δ-system lemma, there is a $K \subseteq \theta_2$ of size θ_2 such that $\langle \operatorname{dom} p_\alpha : \alpha \in K \rangle$ forms a Δ-system with root R, even more, we may assume that the Δ-system is quite uniform, i.e., all such domains have the same size and, for every $\xi \in R$, if $\xi \in E$ then all the $(s^\alpha_\xi, m^\alpha_\xi)$ for $\alpha \in K$ are equal to a single (s^*_ξ, m^*_ξ), else, if $\xi \notin E$ then all $p_\alpha(\xi)$ for $\alpha \in K$ are equal to a single $q(\xi)$ (because $\dot{\mathbb{Q}}_\xi$ is forced to have size $< \kappa$). To conclude the θ_2-Fr-Knaster property, the point is to ensure that, for any countable subsequence $\bar{p} = \langle p_n : n < \omega \rangle$ of $\langle p_\alpha : \alpha \in K \rangle$ there is a q that forces $\dot{W}(\bar{p})$ infinite. As $\bar{p}$ also forms a uniform Δ-system, it would be suitable to construct q with domain R. We already have $q(\xi)$ for $\xi \in R \smallsetminus E$. For each $\xi \in R \cap E$, $\langle p_n(\xi) : n < \omega \rangle$ is a sequence of $\mathbb{P}'_\xi$-names in $\mathbb{E}(s^*_\xi, m^*_\xi)$, so we may expect that $q(\xi)$ is defined as some ultrafilter limit of that sequence.

The main challenge now is to construct the ultrafilters for $\xi \in E$ that will serve to calculate the desired ultrafilter limits. One problem is that a single sequence of (names of) ultrafilters indexed by E may not work to ensure the θ_2-Fr-Knaster property, and actually it is necessary to guarantee that only $< \lambda$-many such sequences of ultrafilters will be enough. The reason is that $\dot{\mathbb{Q}}_\xi = \mathbb{E}^{V^{\mathbb{P}'_\xi}}$ should be constructed such that ultrafilter limits are available within this poset for the ξ-th ultrafilters in all the sequences. This means that $\mathbb{P}'_\xi$ should be constructed such that the restriction to $V^{\mathbb{P}'_\xi}$ of the ξ-th ultrafilters in all the sequences are in $V^{\mathbb{P}'_\xi}$. Thanks to Theorem 4.2, it can be guaranteed that just μ-many sequences of ultrafilters are enough, so the construction of $\mathbb{P}_\delta$ and the proof of Theorem 4.1 can be concluded. Note that this dynamic indicates that the sequences of ultrafilters and the FS iteration must be constructed simultaneously by recursion.

Though details on the proof of Theorem 4.1 are omitted, we roughly sketched them just to picture the complexity of such argument. Now, once we aim to do something similar to conclude the proof of Theorem 3.9, we realize that we have to deal with a matrix of ultrafilters of dimensions $\nu \times \pi$, which may look much more complicated than the argument of Theorem 4.1. However, when trying to work out with the construction already defined in the first part of the proof of Theorem 3.9, there is the advantage that we *only* need to find a suitable matrix of ultrafilters instead of constructing it simultaneously with the forcing as in the previous sketch. This is in fact the case, and what is the most surprising is that Theorem 4.2 is not necessary since it is enough to construct one matrix of ultrafilters for each countable uniform Δ-system.

Lemma 4.8 ([BCM]). *Let $\mathbb{P}_{\nu,\pi}$ be the 2D coherent system constructed in the first part of the proof of Theorem 3.9. If $\bar{p} = \langle p_n : n < \omega\rangle$ is a uniform Δ-system in $\mathbb{P}_{\nu,\pi}$ then there is a $q \in \mathbb{P}_{\nu,\pi}$ such that $q \Vdash$ "$\dot{W}_{\mathbb{P}_{\nu,\pi}}(\bar{p})$ is infinite". Moreover, if D is a non-principal ultrafilter in the ground model then there is some $q \in \mathbb{P}_{\nu,\pi}$ that forces $\dot{W}_{\mathbb{P}_{\gamma,\pi}}(\bar{p}) \in D^+$.*

Second part of the proof of Theorem 3.9. By a Δ-system argument, Lemma 4.8 implies that $\mathbb{P}_{\nu,\pi}$ is θ_2-Fr-Knaster. Even more, for any uncountable regular κ between θ_2 and λ, the Cohen reals added by $\mathbb{C}_\kappa \lessdot \mathbb{P}_{0,1}$ form a strongly κ-ω^ω-unbounding family in $V_{\nu,\pi}$. In particular, $\mathbb{P}_{\nu,\pi}$ forces $\mathfrak{b} \leq \theta_2$ and $\lambda \leq \mathfrak{d}$, which concludes the proof. □

We conclude this section with one application of Theorem 3.9. Very recently, Goldstern, Kellner and Shelah [GKS] proved that, *assuming four strongly compact cardinals*, there is a ccc poset that divides Cichoń's diagram into 10 different values, which is the maximal number of values the diagram can be divided. First, a poset that separates the left side of Cichoń's diagram is obtained, e.g. the one in Theorem 4.1, and then Boolean ultrapowers are applied to this poset to force that the right side can be separated in addition. For this purpose, the poset for Theorem 4.1 is modified so that it is possible to construct it under GCH.

Theorem 4.9 ([GKS]). *Assume GCH and that $\theta_0 < \theta_1 < \theta_2 < \theta_3 < \lambda_4$ are uncountable regular strongly $\aleph_1$-inaccessible cardinals ([3]) and that θ_2 is a successor of a regular cardinal. Then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \theta_2$, $\mathrm{non}(\mathcal{M}) = \theta_3$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_4$.*

Theorem 4.10 ([GKS]). *Assume GCH and that $\kappa_8 < \theta_0 < \kappa_7 < \theta_1 < \kappa_6 < \theta_2 < \kappa_5 < \theta_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8$ are uncountable regular cardinals such that θ_i (i=0,1,2,3) and λ_4 are strongly $\aleph_1$-inaccessible cardinals, θ_2 is a successor of a regular cardinal and each κ_j $(j = 5,6,7,8)$ is a strongly compact cardinal. Then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \theta_2$, $\mathrm{non}(\mathcal{M}) = \theta_3$, $\mathrm{cov}(\mathcal{M}) = \lambda_4$, $\mathfrak{d} = \lambda_5$, $\mathrm{non}(\mathcal{N}) = \lambda_6$, $\mathrm{cof}(\mathcal{N}) = \lambda_7$ and $\mathfrak{c} = \lambda_8$.*

We summarize very briefly the technique of Boolean ultrapowers from [KTT, GKS]. Assume GCH (for simplicity), let κ be a strongly compact cardinal and $\lambda > \kappa$ regular. Consider the Boolean completion $\mathbb{B}_{\kappa,\lambda}$ of the poset $\mathrm{Fn}_{<\kappa}(\lambda,\kappa)$ of partial functions from λ to κ with domain of size $< \kappa$ (ordered by $\supseteq$). It is possible to find a suitable κ-complete ultrafilter U on $\mathbb{B}$ such that its corresponding elementary embedding $j : V \to M$ satisfies (among other properties):

(1) j has critical point κ, $\mathrm{cf}(j(\kappa)) = \lambda$ and $\lambda \leq j(\kappa) < \lambda^+$;

(2) if $|A| < \kappa$ then $j[A] = j(A)$;

(3) if $\theta > \kappa$ is regular then $\max\{\lambda,\theta\} \leq j(\theta) < \max\{\lambda,\theta\}^+$;

(4) if $\theta > \kappa$ and I is a $< \theta$-directed partial order then $j[I]$ is cofinal in $j(I)$.

[3]Recall that θ is a *strongly κ-inaccessible cardinal* if, for any cardinals $\mu < \theta$ and $\nu < \kappa$, $\mu^\nu < \theta$.

As a consequence,

Lemma 4.11 ([KTT, GKS]). *Additionally to the above, assume that* $\mathbf{A}$ *is an analytic relational system and that* $\mathbb{P}$ *is a ccc poset. Then:*

(a) $j(\mathbb{P})$ *is ccc (in* V*, not just in* M*).*

(b) If $\theta \neq \kappa$ *is regular and* $\mathbb{P}$ *adds a strongly* θ-$\mathbf{A}$*-unbounded family of size* θ*, then* $j(\mathbb{P})$ *also adds a strongly* θ-$\mathbf{A}$*-unbounded family of size* θ*.*

(c) If $\mathbb{P}$ *adds a strongly* θ-$\mathbf{A}$ *dominating family of size* λ' *then*

  (i) whenever $\theta < \kappa$*,* $j(\mathbb{P})$ *adds a strongly* θ-$\mathbf{A}$ *dominating family of size* $|j(\lambda')|$*;*

  (ii) whenever $\kappa < \theta$*,* $j(\mathbb{P})$ *adds a strongly* θ-$\mathbf{A}$ *dominating family of size* λ'*.*

Thanks to the previous lemma, it is enough to define the poset for Theorem 4.10 as $j_9(j_8(j_7(j_6(\mathbb{P}))))$ where $\mathbb{P}$ is the poset from Theorem 4.9 and $j_i : V \to M_i$ $(i = 5, 6, 7, 8)$ is the elementary embedding corresponding to a suitable Boolean ultrapower by $\mathbb{B}_{\kappa_i,\lambda_i}$.

Note that the poset obtained in Theorem 3.9 can be constructed directly under GCH and it separates the cardinals of the left of Cichoń's diagram and, in addition, separates $\mathrm{cov}(\mathcal{M})$ on the right side. Therefore, applying the aforementioned Boolean ultrapower techniques to this poset, we can actually separate Cichoń's diagram into 10 values by using *three strongly compact cardinals.* Furthermore, the hypothesis on the regular cardinals of Theorem 4.10 can also be relaxed.

Theorem 4.12 ([BCM]). *Assume GCH and that* $\kappa_8 < \theta_0 < \kappa_7 < \theta_1 < \kappa_6 < \theta_2 \leq \theta_3 \leq \lambda_4 \leq \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8$ *are uncountable regular cardinals such that each* κ_j $(j = 6, 7, 8)$ *is a strongly compact cardinal. Then there is a ccc poset that forces* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \theta_2$, $\mathrm{non}(\mathcal{M}) = \theta_3$, $\mathrm{cov}(\mathcal{M}) = \lambda_4$, $\mathfrak{d} = \lambda_5$, $\mathrm{non}(\mathcal{N}) = \lambda_6$, $\mathrm{cof}(\mathcal{N}) = \lambda_7$ *and* $\mathfrak{c} = \lambda_8$*.*

If $\mathbb{P}$ is the ccc poset obtained in Theorem 3.9 (with $\mu = \theta_3$, $\nu = \lambda_4$ and $\lambda = \lambda_5$), then $j_8(j_7(j_6(\mathbb{P})))$ is the desired poset for the previous theorem.

Without using large cardinals, the maximum number of different cardinals in Cichoń's diagram that have been obtained so far is 7 and all such examples belong to [FFMM18, Mej, BCM] (e.g. Theorems 3.9 and 3.10). It is still unknown how to force 8 different values modulo ZFC alone.

Acknowledgements

The author is very grateful with professor Jörg Brendle for inviting him to deliver the minicourse of the RIMS 2017 Set Theory Workshop on Iterated Forcing and Cardinal Invariants, and also for organizing such a wonderful conference. The lectures and the production of his manuscript was supported by professor Brendle and also by Kyoto University RIMS.

References

[Bar88] Tomek Bartoszyński. On covering of real line by null sets. *Pacific J. Math.*, 131(1):1–12, 1988.

[BCM] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. Filter-linkedness and its effect on the preservation of cardinal characteristics. In preparation.

[BF11] Jörg Brendle and Vera Fischer. Mad families, splitting families and large continuum. *J. Symbolic Logic*, 76(1):198–208, 2011.

[BJ95] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line.* A K Peters, Wellesley, Massachusetts, 1995.

[BM14] Jörg Brendle and Diego Alejandro Mejía. Rothberger gaps in fragmented ideals. *Fund. Math.*, 227(1):35–68, 2014.

[Bre91] Jörg Brendle. Larger cardinals in Cichoń's diagram. *J. Symbolic Logic*, 56(3):795–810, 1991.

[Bre02] Jörg Brendle. Mad families and iteration theory. In *Logic and algebra*, volume 302 of *Contemp. Math.*, pages 1–31. Amer. Math. Soc., Providence, RI, 2002.

[Bre03] Jörg Brendle. The almost-disjointness number may have countable cofinality. *Trans. Amer. Math. Soc.*, 355(7):2633–2649 (electronic), 2003.

[Bre05] Jörg Brendle. Templates and iterations, Luminy 2002 lecture notes. *Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku*, (1423):1–12, 2005.

[BS89] Andreas Blass and Saharon Shelah. Ultrafilters with small generating sets. *Israel J. Math.*, 65(3):259–271, 1989.

[CM] Miguel A. Cardona and Diego A. Mejía. On cardinal characteristics of Yorioka ideals. Submitted.

[EK65] Ryszard Engelking and Monika Karłowicz. Some theorems of set theory and their topological consequences. *Fund. Math.*, 57:275–285, 1965.

[FFMM18] Vera Fischer, Sy D. Friedman, Diego A. Mejía, and Diana C. Montoya. Coherent systems of finite support iterations. *J. Symbolic Logic*, pages 1–29, 2018.

[FM17] Vera Fischer and Diego Alejandro Mejía. Splitting, bounding, and almost disjoitness can be quite different. *Canad. J. Math*, 69:502–531, 2017.

[GKS] Martin Goldstern, Jakob Kellner, and Saharon Shelah. Cichoń's maximum. Preprint.

[GMS16] Martin Goldstern, Diego Alejandro Mejía, and Saharon Shelah. The left side of Cichoń's diagram. *Proc. Amer. Math. Soc.*, 144(9):4025–4042, 2016.

[GS93] Martin Goldstern and Saharon Shelah. Many simple cardinal invariants. *Arch. Math. Logic*, 32(3):203–221, 1993.

[JS90] Haim Judah and Saharon Shelah. The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing). *J. Symbolic Logic*, 55(3):909–927, 1990.

[JS93] Haim Judah and Saharon Shelah. Adding dominating reals with the random algebra. *Proc. Amer. Math. Soc.*, 119(1):267–273, 1993.

[KO14] Shizuo Kamo and Noboru Osuga. Many different covering numbers of Yorioka's ideals. *Arch. Math. Logic*, 53(1-2):43–56, 2014.

[KS12] Jakob Kellner and Saharon Shelah. Creature forcing and large continuum: the joy of halving. *Arch. Math. Logic*, 51(1-2):49–70, 2012.

[KTT] Jakob Kellner, Anda Tănasie, and Fabio Tonti. Compact cardinals and eight values in Cichoń's diagram. Preprint.

[Mej] Diego A. Mejía. Matrix iterations with vertical support restrictions. Submitted.

[Mej13] Diego Alejandro Mejía. Matrix iterations and Cichon's diagram. *Arch. Math. Logic*, 52(3-4):261–278, 2013.

[Mej15] Diego A. Mejía. Template iterations with non-definable ccc forcing notions. *Ann. Pure Appl. Logic*, 166(11):1071–1109, 2015.

[Mil81] Arnold W. Miller. Some properties of measure and category. *Trans. Amer. Math. Soc.*, 266(1):93–114, 1981.

[Paw92] Janusz Pawlikowski. Adding dominating reals with ω^ω bounding posets. *J. Symbolic Logic*, 57(2):540–547, 1992.

[She00] Saharon Shelah. Covering of the null ideal may have countable cofinality. *Fund. Math.*, 166(1-2):109–136, 2000. Saharon Shelah's anniversary issue.

[She04] Saharon Shelah. Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing. *Acta Math.*, 192(2):187–223, 2004.